NORTH-HOLLAND

Linear Operators Which Preserve Sign-Nonsingular Matrices

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Dedicated to Professor John Maybee on the occasion of his 65th birthday.

Submitted by Hans Schneider

ABSTRACT

We characterize the linear operators that preserve the set of sign-nonsingular matrices. We show that if a linear operator T preserves the set of sign-nonsingular matrices, then T preserves the set of matrices of term rank 1. Then by a result of Beasley and Pullman on preservers of matrices of term rank 1, we can obtain the structure of T.

1. INTRODUCTION

Let $\mathbb{K}$ be a field, and $\mathcal{M} = \mathcal{M}_{m,n}(\mathbb{K})$ be the set of all $m \times n$ matrices over $\mathbb{K}$. If T is a linear operator on $\mathcal{M}$ and $\mathcal{K}$ is a subset of $\mathcal{M}$, then T *preserves* $\mathcal{K}$ if $T(X) \in \mathcal{K}$ for each X in $\mathcal{K}$. If f is a function on $\mathcal{M}$, then T *preserves* f if $f(T(A)) = f(A)$ for all A in $\mathcal{M}$.

A common problem considered in linear algebra is called a preserver problem, that is, characterize those linear operators which "preserve" a function or a set. The classification of preservers began about 100 years ago. In 1897, Frobenius [3] characterized the linear operators on $\mathcal{M}$ which pre-

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serve certain matrix functions: those linear operators on $\mathcal{M}$ that preserve the determinant and those that preserve the characteristic polynomial.

After half a century of relative inactivity there was renewed interest in preserver problems. That interest was sparked by the investigation of rank preservers in 1959 by Marcus and Moyls [5]. They proved: If $\mathbb{F}$ is an algebraically closed field of characteristic 0 and T is a rank preserver, then there exist $m \times m$ and $n \times n$ matrices U and V, respectively, such that either

$$T(A) = UAV \qquad \text{for all} \quad A \in \mathcal{M}_{m,n}(\mathbb{F}) \tag{1.1}$$

or

$$m = n \quad \text{and} \quad T(A) = UA^{\mathrm{t}}V \qquad \text{for all} \quad A \in \mathcal{M}_{m,n}(\mathbb{F}). \tag{1.2}$$

We call operators of the type (1.1) and (1.2) (U, V) *operators.*

Also in 1959, Marcus and Moyls [6] found that T is a rank preserver if and only if T is a rank-1 preserver, that is, T preserves the set of matrices whose rank is 1. In 1967, Westwick [8] generalized these results to matrices over arbitrary algebraically closed fields.

Characterizations of preservers have been appearing regularly over the past twenty years, and an excellent summary of nearly all characterizations of linear preservers can be found in a special issue of *Linear and Multilinear Algebra,* edited by S. Pierce with input by leaders in the field [7]. This article includes a list of over 200 articles written on the subject.

Within the past two or three years, structural matrix theory has been an active area of reasearch in pure and applied mathematics. In structural matrix theory, one is concerned only with the locations of the zero and nonzero entries (and perhaps the sign of the nonzero entries), and not with the magnitude of the entries. At the core of this research is the area of sign-nonsingular matrices, $n \times n$ real matrices such that every $n \times n$ real matrix which has the same $(0, +, -)$ sign pattern is nonsingular.

In this paper, we will characterize the linear opeators that preserve the set of sign-nonsingular matrices. We show that if a linear opertor T preserves the set of sign-nonsingular matrices, then T preserves the set of matrices of term rank 1. Then by [1], we can obtain the structure of T.

2. PRELIMINARIES

Henceforth, we restrict our matrices to $n \times n$ real matrices, $\mathcal{M} = \mathcal{M}_{n,n}$ $(\mathbb{R})$.

The number of nonzero entries in a matrix A is denoted $|A|$, the number of elements in a set S is denoted $|S|$, and the absolute value of a real number is denoted $|a|$. There should be no confusion, as the type of mathematical object is evident from the context.

If S is a set and $\mathcal{S}$ is the space spanned by S, then we denote $\mathcal{S} = \langle S \rangle$.

We denote the *Hadamard product* of $A = (a_{i,j})$ and $B = (b_{i,j})$ in $\mathcal{M}_{n,m}$ by $A \circ B$. That is, $A \circ B = (a_{i,j} b_{i,j})$. If $A = (a_{i,j})$ and $B = (b_{i,j})$ are in $\mathcal{M}_{n,m}$, we say that B *dominates* A (written $B \geq A$ or $A \leq B$) if $b_{i,j} = 0$ implies $a_{i,j} = 0$ for all i, j.

Let $T : \mathcal{M}_{n,m} \to \mathcal{M}_{n,m}$ be a linear operator. We say T preserves the subset $\mathcal{X}$ of $\mathcal{M}_{n,m}$ if T maps each matrix in the set $\mathcal{X}$ to a matrix in $\mathcal{X}$. We say T *strongly preserves* the subset $\mathcal{X}$ of $\mathcal{M}_{n,m}$ if T preserves both $\mathcal{X}$ and $\mathcal{M}_{n,m} \setminus \mathcal{X}$, the complement of $\mathcal{X}$ in $\mathcal{M}_{n,m}$. We call such T an $\mathcal{X}$ *preserver* or an $\mathcal{X}$ *strong preserver*, respectively.

The *term rank* is the minimum number, $t(A)$, of lines (columns or rows) which contain all nonzero entries of A.

Let $A = (a_{i,j})$. We denote by $A[\alpha \mid \beta]$ the submatrix of A on rows α and columns β.

We end this section with some results which are heavily used throughout this document.

THEOREM 2.1 (Beasley and Pullman [1, Corollary 3.1.2.]). *Suppose that T is a nonsingular linear operator on $\mathcal{M}$. The operator preserves the set of matrices of term rank* 1 *if and only if T is one of or a composition of some of the following operators:*

(i) $X \to X^t$.
(ii) $X \to PXQ$ *for some fixed but arbitrary permutation matrices P and Q in $\mathcal{M}$.*
(iii) $X \to A \circ X$ *for some fixed but arbitrary matrix A in $\mathcal{M}$ with no zero entries.*

A sign-nonsingular matrix is an $n \times n$ real matrix A such that every $n \times n$ real matrix which has the same $(0, +, -)$ sign pattern as A is nonsingular. The study of sign-nonsingular matrices is related to the problem of finding a way to change the signs in a matrix in such a way that the permanent of the first is the determinant of the second. The next lemma shows that the set of sign-nonsingular matrices is precisely the set of nonsingular matrices that can be so signed.

LEMMA 2.1 (See Brualdi, Chavey, and Shader [2]). *A matrix $A \in \mathcal{M}$ is a sign-nonsingular matrix if and only if there is a nonzero term in its determinant expansion and every nonzero term has the same sign.*

The next lemma is regularly used in any study of sign-nonsingularity.

LEMMA 2.2 (Gibson [4]). *If $A \in \mathcal{M}$ is a sign-nonsingular matrix, then* $|A| \leq (n^2 + 3n - 2)/2$.

It follows from Lemma 2.2 that no 3×3 sign-nonsingular matrix can have all nonzero entries. Thus, no sign-nonsingular matrix can have three rows with all nonzero entries. This fact will be used in the proof of Lemma 3.5.

If $A \in \mathcal{M}_n$ and $B \in \mathcal{M}_n$ are sign-nonsingular matrices, then $A \circ B$ need not be a sign-nonsingular matrix. For example,

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

are sign-nonsingular matrices. But $A \circ B = O$ is not a sign-nonsingular matrix. However, a direct application of the definition of sign-nonsingular matrices establishes the following lemma.

LEMMA 2.3. *If $A, M = (m_{i,j}) \in \mathcal{M}$ with $m_{i,j} > 0$ (or $m_{i,j} < 0$) for all (i, j), then $A \circ M$ is a sign-nonsingular matrix if and only A is a sign-nonsingular matrix.*

If $A, B \in \mathcal{M}_m$ are sign-nonsingular matrices, then AB need not be a sign-nonsingulr matrix. For example, if

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & -1 \\ 2 & 2 \end{pmatrix},$$

then A and B are sign-nonsingular. But

$$AB = \begin{pmatrix} -1 & -3 \\ 3 & 1 \end{pmatrix}$$

is not sign-nonsingular, since the matrix

$$C = \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix},$$

which has the same $(0, +, -)$ sign pattern as AB, is singular. However, we have the following lemmas which are easily proven.

LEMMA 2.4. *If $A \in \mathcal{M}$ is a sign-nonsingular matrix and $P \in \mathcal{M}$ is a permutation matrix, then AP and PA are also sign-nonsingular matrices.*

LEMMA 2.5. *If $A \in \mathcal{M}$ is a sign-nonsingular matrix and $D \in \mathcal{M}$ is a diagonal matrix with all nonzero entries on its main diagonal, then AD and DA are also sign-nonsingular matrices.*

3. SIGN-NONSINGULAR PRESERVERS

In this section we will investigate the linear operators that preserve sign-nonsingular matrices. We will prove that if $n \geq 3$ and $T : \mathcal{M} \to \mathcal{M}$ is a linear operator that preserves sign-nonsingular matrices, then for any matrix $X \in \mathcal{M}$, $T(X) = P_1 S_1 (X \circ M) S_2 P_2$, where $P_i \in \mathcal{M}$ $(i = 1, 2)$ is a permutation matrix, $S_i \in \mathcal{M}$ $(i = 1, 2)$ is a diagonal matrix of ± 1's, and $M = (m_{i,j}) \in \mathcal{M}$ with $m_{i,j} > 0$. We then summarize our investigation of linear operators that preserve sign-nonsingular matrices for $n = 2$. Henceforth we assume that $n \geq 3$.

A $n \times m$ matrix with only one nonzero entry, say the (i, j)th entry, $E_{i,j}$, is called a cell. We denote the set of cells whose ith row is nonzero by $\mathcal{R}_i$, i.e. $\mathcal{R}_i = \{E_{i,1}, E_{i,2}, \ldots, E_{i,n}\}$, and the set of cells whose jth column is nonzero by $\mathcal{C}_j = \{E_{1,j}, E_{2,j}, \ldots, E_{n,j}\}$. Let $R_i = \sum_{j=1}^{n} E_{i,j}$ and $C_j = \sum_{i=1}^{n} E_{i,j}$. That is, R_i is the matrix whose ith row is all ones and all other entries are zero.

Let $T : \mathcal{M} \to \mathcal{M}$ be a linear operator that preserves sign-nonsingular matrices. Define linear operators $\mathbf{T}_i$ and $\mathbf{T}^j$ on $\mathcal{M}$ by $\mathbf{T}_i(X) = \mathbf{T}(X) \circ R_i$ and $\mathbf{T}^j(X) = \mathbf{T}(X) \circ C_j$. Let

$$\begin{aligned} U_i &= \{\mathbf{T}_i(E_{k,l}) : 1 \leq k, l \leq n\}, & \mathcal{U}_i = \langle U_i \rangle, \\ V_j &= \{\mathbf{T}^j(E_{k,l}) : 1 \leq k, l \leq n\}; & \mathcal{V}_i = \langle V_i \rangle, \\ W_i &= \{E_{k,l} : \mathbf{T}_i(E_{k,l}) \neq O\}, & \\ X_j &= \{E_{k,l} : \mathbf{T}^j(E_{k,l}) \neq O\}. & \end{aligned}$$

The purpose of the next seven lemmas is to show that $W_i = \mathcal{R}_{\sigma(i)}$ and $X_j = \mathcal{C}_{\tau(j)}$ for all i, j, $1 \leq i, j \leq n$, for some permutations σ and τ of $\{1, 2, \ldots, n\}$.

LEMMA 3.1. *For each i and j, $1 \leq i, j \leq n$, $\dim \mathcal{U}_i = n$, $\dim \mathcal{V}_j = n$, and $|W_i|, |X_j| \geq n$.*

Proof. It is evident that $\dim \mathcal{U}_i \leq n$. If $\dim \mathcal{U}_i < n$, then there exist $r <$

n cells in W_i whose images generate $\mathcal{U}_i$. Suppose $\{E_1, E_2, \ldots, E_r\} \subseteq W_i$ such that $\mathcal{U}_i = \langle\{T_i(E_1), T_i(E_2), \ldots, T_i(E_r)\}\rangle$. Then, since $r < n$, for some permutation matrices $P, Q \in \mathcal{M}$, $P(\sum_{i=1}^{r} E_i)Q$ is an upper triangular matrix with all zeros on its main diagonal. Hence $I_n + P(\sum_{i=1}^{r} \alpha_i E_i)Q$ is a sign-nonsingular matrix for any α_i's, and so is $P^t[I_n + P(\sum_{i=1}^{r} \alpha_i E_i)Q]Q^t = P^t Q^t + \sum_{i=1}^{r} \alpha_i E_i$. Therefore $T(P^t Q^t + \sum_{i=1}^{r} \alpha_i E_i)$ must be a sign-nonsingular matrix for any choice of α_i's. But for some choice of α_i's, $T(P^t Q^t + \sum_{i=1}^{r} \alpha_i E_i)$ has a zero ith row and hence is not sign-nonsingular. This contradiction implies that $\dim \mathcal{U}_i = n$. If $|W_i| < n$, then $\dim \mathcal{U}_i < n$, a contradiction. The proof is complete. ∎

LEMMA 3.2. *Let $1 \leq i, j \leq n$. If $\{E_1, E_2, \ldots, E_n\} \subseteq W_i$ $[\subseteq X_j]$ such that $\mathcal{U}_i = \langle\{T_i(E_1), T_i(E_2), \ldots, T_i(E_n)\}\rangle$ $[\mathcal{V}_j = \langle\{T^j(E_1), T^j(E_2), \ldots, \langle T^j(E_n)\}\rangle]$, then for each r, $1 \leq r \leq n$, there is an $E_k \in W_i$ $[\in X_j]$ whose nonzero entry is in row r, or for each r, $1 \leq r \leq n$, there is an $E_k \in W_i[\in X_j]$ whose nonzero entry is in column r.*

Proof. If not, then there exists $\{E_1, E_2, \ldots, E_n\} \subseteq W_i$ such that $\mathcal{U}_i = \langle\{T_i(E_1), T_i(E_2), \ldots, T_i(E_n)\}\rangle$ and $\sum_{i=1}^{n} E_i$ has a zero row and a zero column. By permuting rows and/or columns, we have, without loss of generality, that $\sum_{i=1}^{n} E_i$ is an upper triangular matrix with all zeros on its main diagonal and $\langle\{T_i(E_1), T_i(E_2), \ldots, T_i(E_n)\}\rangle = \mathcal{U}_i$. Hence $I_n + \sum_{i=1}^{n} \alpha_i E_i$ is a sign-nonsingular matrix for any α_i's. Therefore $T(I_n + \sum_{i=1}^{n} \alpha_i E_i)$ must be a sign-nonsingular matrix for any choice of α_i's. But for some choice of α_i's, $T(I_n + \sum_{i=1}^{n} \alpha_i E_i)$ has a zero ith row and hence is not sign-nonsingular, a contradiction. ∎

LEMMA 3.3. *For any $\mathcal{U}_i$ $[\mathcal{V}_j]$ there exists k such that either $\mathcal{R}_k \subseteq W_i$ $[\mathcal{R}_k \subseteq X_j]$ or $\mathcal{C}_k \subseteq W_i$ $[\mathcal{C}_k \subseteq X_j]$ and $\langle T_i(\mathcal{R}_k)\rangle = \mathcal{U}_i$ $[\langle T^j(\mathcal{R}_k)\rangle = \mathcal{V}_j]$ or $\langle T_i(\mathcal{C}_k)\rangle = \mathcal{U}_i[\langle T^j(\mathcal{C}_k)\rangle = \mathcal{V}_j]$.*

Proof. Suppose $\mathcal{E} = \{E_1, E_2, \ldots, E_n\} \subseteq W_i$ and $\langle T_i(\mathcal{E})\rangle = \mathcal{U}_i$. By Lemma 3.2, we can assume, without loss of generality, that for each k, E_k has its nonzero entry in column k. Further we can assume, by permuting rows and/or columns, that $E_1 + \cdots + E_n$ is an upper triangular matrix, with possibly nonzero entries on the main diagonal. ∎

Case 1. $E_1 + E_2 + \cdots + E_n$ has a zero row. Since $E_1 + \cdots + E_n$ is an upper triangular matrix and E_1 has its nonzero entry in the first column, we have that $E_1 = E_{1,1}$. Suppose $E_{1,k} \notin W_i$. By permuting rows and columns, we can assume that $E_{1,n} \notin W_i$. Let

$$D_\alpha = E_{2,1} + E_{3,2} + \cdots + E_{n,n-1} + \alpha E_{1,n} + \sum_{j=1}^{n} \beta_j E_j,$$

where the β_j are chosen so that $T_i(D_\alpha) = 0$ for all choices of α. Expanding the determinant of D_α on the nth column we have that

$$\begin{aligned} \det D_\alpha &= (-1)^{n+1}\alpha \det D_\alpha[2,\ldots,n \mid 1,\ldots,n-1] \\ &\quad + (-1)^{n+k}\beta_n \det D_\alpha[1,\ldots,\widehat{k},\ldots,n \mid 1,\ldots,n-1], \end{aligned}$$

where $E_n = E_{k,n}$. Note here that since $E_{1,n} \notin W_i$, D_α has exactly two possibly nonzero entries in the nth column, α in the $(1,n)$ position and β_n in the (k,n) position. Now

$$D_\alpha[2,\ldots,n \mid 1,\ldots,n-1] = \begin{bmatrix} 1 & & & & * \\ 0 & \cdot & & & \\ \cdot & & \cdot & & \\ \cdot & & & \cdot & \\ 0 & \cdot & \cdot & 0 & 1 \end{bmatrix},$$

so $\det D_\alpha[2,\ldots,n \mid 1,\ldots,n-1] = 1$. Also

$$\begin{aligned} &D_\alpha[1,\ldots,\widehat{k},\ldots,n \mid 1,\ldots,n-1] \\ &= \left[\begin{array}{ccccc|ccccc} \beta_1 & & & & & & & & & \\ 1 & & * & & & & & & & \\ 0 & \cdot & & & & & & * & & \\ \cdot & & \cdot & & & & & & & \\ \cdot & & & \cdot & & & & & & \\ 0 & \cdot & \cdot & 0\ 1 & * & & & & & \\ \hline & & & & & 1 & & & * & \\ & & & & & & \cdot & & & \\ & & 0 & & & & & \cdot & & \\ & & & & & 0 & & & \cdot & \\ & & & & & & & & & 1 \end{array}\right] \end{aligned}$$

$$= \left[\begin{array}{c|ccccc} D_\alpha[1,\ldots,k-1 \mid 1,\ldots,k-1] & & & * & & \\ \hline & 1 & & & * & \\ & & \cdot & & & \\ 0 & & & \cdot & & \\ & & & & \cdot & \\ & 0 & & & & \cdot \\ & & & & & 1 \end{array}\right].$$

That is,

$$\det D_\alpha[1,\ldots,\widehat{k},\ldots,n \mid 1,\ldots,n-1] = \det D_\alpha[1,\ldots,k-1 \mid 1,\ldots,k-1].$$

It follows that

$$\det D_\alpha = (-1)^{n+1}\alpha + (-1)^{n+k}\beta_n \det D_\alpha[1,\ldots,k-1 \mid 1,\ldots,k-1].$$

Further, as above,

$$\det D_\alpha[1,\ldots,k-1 \mid 1,\ldots,k-1] = (-1)^{k-1+l}\beta_{k-1} \det D_\alpha[1,\ldots,l \mid 1,\ldots,l]$$

for some $l < k-1$. Continuing, we have that

$$\det D_\alpha[1,\ldots,\widehat{k},\ldots,n \mid 1,\ldots,n-1] = \pm\beta_{k-1}\beta_{l-1}\cdots,$$

a single term. That is,

$$\det D_\alpha = (-1)^{n+1}\alpha \pm \beta_n\beta_{k-1}\beta_{l-1}\cdots.$$

Choosing α such that both terms in the expansion of $\det D_\alpha$ have the same sign, or $\alpha = 1$ if $\beta_n\beta_{k-1}\beta_{l-1}\cdots = 0$, we have that D_α is sign-nonsingular. But $T_i(D_\alpha) = 0$, so $T(D_\alpha)$ has a zero ith row, contradicting that T preserves sign-nonsingular matrices. Thus $E_{1,n} \in W_i$, and in fact, $E_{1,k} \in W_i$ for all k. Now, if $T_i(E_{1,n})$ and $T_i(E_n) = T_i(E_{k,n})$ are linearly independent, then $\{T_i(E_{1,n}), T_i(E_{k,n}), T_i(E_1), \ldots, T_i(E_{n-1})\}\backslash T_i(E_l)$ for some $l \neq n$ spans $\mathcal{U}_i$, but $\{E_{1,n}, E_{k,n}, E_1, \ldots, E_{n-1}\}\backslash E_l$ all have zero entries in column l and in the same row as $\{E_1, \ldots, E_n\}$. This contradicts Lemma 3.2. Thus $T_i(E_{1,n})$ and $T_i(E_{k,n})$ are dependent. It now follows that

$$\{T_i(E_1), T_i(E_2), \ldots, T_i(E_n)\} = \{T_i(E_{1,1}), \gamma_2 T_i(E_{1,2}), \ldots, \gamma_n T_i(E_{1,n})\}.$$

Thus $\langle T_i(\mathcal{R}_1)\rangle = \mathcal{U}_i$.

Case 2. $E_1 + E_2 + \cdots + E_n = I_n$. Suppose $E_{k,l} \notin W_i$ for some pair (k,l). By permuting rows and columns we have, without loss of generality, that $E_{1,n} \notin W_i$. As above, let

$$D_\alpha = E_{2,1} + E_{3,2} + \cdots + E_{n,n-1} + \alpha E_{1,n} + \sum_{j=1}^{n} \beta_j E_{j,j},$$

where the β_j's are chosen such that $T_i(D_\alpha) = 0$. Then

$$\det D_\alpha = (-1)^{n+1}\alpha + \beta_1\beta_2 \cdots \beta_n.$$

Choosing α such that both terms in the expansion of $\det D_\alpha$ have the same sign, or $\alpha = 1$ if $\beta_1\beta_2 \cdots \beta_n = 0$, we have that D_α is sign-nonsingular. But $T(D_\alpha)$ has zero ith row, a contradiction. Thus $E_{k,l} \in W_i$ for all (k,l). Now, if $T_i(E_{1,n})$ and $T_i(E_{n,n})$ are dependent, then

$$\{T_i(E_{1,1}), T_i(E_{2,2}), \ldots, T_i(E_{n-1,n-1}), T_i(E_{1,n})\}$$

is a basis for $\mathcal{U}_i$, and $E_{1,1} + \cdots + E_{n-1,n-1} + E_{1,n}$ has zero nth row. If $T_i(E_{1,n})$ and $T_i(E_{n,n})$ are independent, then for some $l \neq n$,

$$\{T_i(E_{1,1}), T_i(E_{2,2}), \ldots, T_i(E_{n,n}), T_i(E_{1,n})\} \backslash T_i(E_{l,l})$$

is a basis for $\mathcal{U}_i$, and $E_{1,1} + E_{2,2} + \cdots + E_{n,n} + E_{1,n} - E_{l,l}$ has zero lth column. In either case, case 1 applies and the theorem follows. ■

The above lemma establishes that the image of a subspace of matrices, each of whose nonzero entries lies on a single fixed row (or column), is also a subspace of matrices, each of whose nonzero entries lies on a single fixed row (or column). We call these spaces row or column spaces. In the next lemma we show that row spaces are mapped to row spaces and column spaces to column spaces, or row spaces are mapped to column spaces and column spaces to row spaces.

LEMMA 3.4. *If* $\langle T_i(\mathcal{R}_k)\rangle = \mathcal{U}_i$ $[\langle T_i(\mathcal{C}_k)\rangle = \mathcal{U}_i]$, *then* $\langle T_r(\mathcal{C}_l)\rangle \neq \mathcal{U}_r$ $[\langle T_r(\mathcal{R}_l)\rangle \neq \mathcal{U}_r]$ *for any* r *and* l.

Proof. If not, then, without loss of generality, assume $\langle T_i(\mathcal{R}_1)\rangle = \mathcal{U}_i$ and $\langle T_r(\mathcal{C}_n)\rangle = \mathcal{U}_r$.

If $i \neq r$, then since for any $\alpha_i \neq 0$ and $\beta_l \neq 0$ the matrix

$$\sum_{i=1}^{n} \alpha_i E_{1,i} + \sum_{l=2}^{n} \beta_l E_{l,n} + \sum_{j=2}^{n-1} E_{j,j}$$

is sign-nonsingular, we can choose $\alpha_i \neq 0$ and $\beta_l \neq 0$ such that

$$T\left(\sum_{i=1}^{n} \alpha_i E_{1,i} + \sum_{l=2}^{n} \beta_l E_{l,n} + \sum_{j=2}^{n-1} E_{j,j}\right)$$

has all positive entries on the ith row and all entries on the rth row of the same sign, either all positive or all negative, a contradiction.

If $i = r$, then $\langle T_i(\mathcal{R}_1)\rangle = \langle T_i(\mathcal{C}_n)\rangle = \mathcal{U}_i$. Now let $j \neq i$. By Lemma 3.3, there exiss $\mathcal{R}_k$ or $\mathcal{C}_k$ such that $\langle T_j(\mathcal{R}_k)\rangle = \mathcal{U}_j$ or $\langle T_j(\mathcal{C}_k)\rangle = \mathcal{U}_j$. If $\langle T_j(\mathcal{R}_k)\rangle = \mathcal{U}_j$ for some k, then since $\langle T_i(\mathcal{C}_n)\rangle = \mathcal{U}_i$ and $i \neq j$, by the above argument we get a contradiction. If $\langle T_j(\mathcal{C}_k)\rangle = \mathcal{U}_j$ for some k, then since $\langle T_i(\mathcal{R}_1)\rangle \neq \mathcal{U}_i$ and $i \neq j$, we get a contradiction again. ■

LEMMA 3.5. *If* $\langle T_i(\mathcal{R}_k)\rangle = \mathcal{U}_i$ $[\langle T_i(\mathcal{C}_k)\rangle = \mathcal{U}_i]$, *then for any* $r \neq i$, $\langle T_r(\mathcal{R}_k)\rangle \neq \mathcal{U}_r$ $[\langle T_r(\mathcal{C}_k)\rangle \neq \mathcal{U}_r]$.

Proof. If not, then, without loss of generality, we may assume that $\langle T_1(\mathcal{R}_1)\rangle = \mathcal{U}_1$ and $\langle T_2(\mathcal{R}_1)\rangle = \mathcal{U}_2$. By Lemmas 3.3 and 3.4, we have that for each $\mathcal{U}_i$ $(i = 3, 4, \ldots, n)$ there exists $\mathcal{R}_{k_i}$ such that $\langle T_i(\mathcal{R}_{k_i})\rangle = \mathcal{U}_i$. Without loss of generality, we assume that $\langle T_3(\mathcal{R}_2)\rangle = \mathcal{U}_3$.

Since $\langle T_1(\mathcal{R}_1)\rangle = \mathcal{U}_1, \langle T_2(\mathcal{R}_1)\rangle = \mathcal{U}_2$, and $\langle T_3(\mathcal{R}_2)\rangle = \mathcal{U}_3$, we can choose $\alpha_{1,j} > 0$ $(1 \leq j \leq n)$, $\alpha_{2,j} > 0$ $(1 \leq j \leq n-1)$, and $\alpha_{2,n} < 0$ such that

$$\left| T_q\left(\sum_{i=1}^{2}\sum_{j=1}^{n} \alpha_{i,j}E_{i,j} - \sum_{k=3}^{n} E_{k,n-k+2}\right)\right| = n, \qquad q = 1, 2, 3.$$

Then $T(\sum_{i=1}^{2}\sum_{j=1}^{n} \alpha_{i,j}E_{i,j} - \sum_{k=3}^{n} E_{k,n-k+2})$ has all nonzero entries in the first three rows. But $\sum_{i=1}^{2}\sum_{j=1}^{n} \alpha_{i,j}E_{i,j} - \sum_{k=3}^{n} E_{k,n-k+2}$ is sign-nonsingular, and therefore $T(\sum_{i=1}^{2}\sum_{j=1}^{n} \alpha_{i,j}E_{i,j} - \sum_{k=3}^{n} E_{k,n-k+2})$ must be sign-nonsingular. But no sign-nonsingular matrix can have three rows with all nonzero entries, a contradiction. ■

LEMMA 3.6. *If* $\langle T_i(\mathcal{R}_k)\rangle = \mathcal{U}_i$ $[\langle T_i(\mathcal{C}_k)\rangle = \mathcal{U}_i]$, *then for any* $l \neq k$, $\langle T_i(\mathcal{R}_l)\rangle \neq \mathcal{U}_i$ $[\langle T_i(\mathcal{C}_l)\rangle \neq \mathcal{U}_i]$.

Proof. If not, then without loss of generality, we may assume that $\langle T_1(\mathcal{R}_1)\rangle = \langle T_1(\mathcal{R}_2)\rangle = \mathcal{U}_1$ and $\langle T_i(\mathcal{R}_{i+1})\rangle = \mathcal{U}_i$ $(2 \leq i \leq n-1)$. Thus by Lemma 3.3, there exists $\mathcal{R}_k$ or $\mathcal{C}_k$ such that $\langle T_n(\mathcal{R}_k)\rangle = \mathcal{U}_n$ or $\langle T_n(\mathcal{C}_k)\rangle = \mathcal{U}_n$. This contradicts Lemmas 3.4 and 3.5. ■

Lemmas 3.5 and 3.6 say that no two distinct row or column spaces have the same image. All that remains is to show that exactly n cells have images with nonzero entries on a fixed row or column.

LEMMA 3.7. *$|W_i| = n$, $|X_j| = n$, and there exists permutations σ and τ such that $W_i = \mathcal{R}_{\sigma(i)}$ and $X_j = \mathcal{C}_{\tau(j)}$ (or vice versa).*

Proof. If $|W_i| \neq n$, then by an argument similar to that in Lemma 3.3, we have that

$$\langle T_i(\mathcal{R}_l)\rangle = \langle\{T_i(E_{l,1}), T_i(E_{l,2}), T_i(E_{l,3}), \ldots, T_i(E_{l,n})\}\rangle = \mathcal{U}_i$$

where $l = 1, 2, \ldots, k$ with $k \geq 2$. This contradicts Lemma 3.6. Hence $|W_i| = n$. Then by Lemma 3.3, $W_i = \mathcal{R}_k$ or $\mathcal{C}_k$ for some k. Also by Lemma 3.4, if $i \neq j$ then $W_i \neq W_j$, and by Lemma 3.5, if $W_i = \mathcal{R}_k$ (or $\mathcal{C}_k$) for some k, then for any i, $W_i \neq \mathcal{C}_l$ (or $\mathcal{R}_l$). Thus there eists permutations σ and τ such that $W_i = \mathcal{R}_{\sigma(i)}$ and $X_j = \mathcal{C}_{\tau(j)}$ (or vice versa). ■

From the above lemmas, we have the following theorem immediately, since any matrix of term rank 1 is an element of $\langle\mathcal{R}_i\rangle$ for some i or some $\langle\mathcal{C}_j\rangle$ for some j.

THEOREM 3.1. *If $n \geq 3$ and $T : \mathcal{M} \to \mathcal{M}$ is a linear operator that preserves sign-nonsingular matrices, then T preserves the set of matrices of term rank 1.*

THEOREM 3.2. *If $n \geq 3$ and $T : \mathcal{M} \to \mathcal{M}$ is a linear operator that preserves sign-nonsingular matrices, then T is one to one on the set of cells.*

Proof. Since T preserves sign-nonsingular matrices, by Theorem 3.1 we have that T preserves the set of matrices of term rank 1. Also by Lemma 3.7, $W_i = \mathcal{R}_{\sigma(i)}$ and $X_j = \mathcal{C}_{\tau(j)}$ (or vice versa). Without loss of generality, we may assume $W_i = \mathcal{R}_i$ $(i = 1, 2, \ldots, n)$ and $X_j = \mathcal{C}_j$ $(j = 1, 2, \ldots, n)$.

First, since T preserves the set of matrices of term rank 1, we have that $T(E) \neq O$ for any cell E.

Also for any i and j, since $E_{i,j} \in W_i \cap X_j$ we have that $T(E_{i,j}) \leq R_i \cap C_j$. That is, $T(E_{i,j})$ is a matrix whose nonzero entries lie in row i and column j. Since $T(E_{i,j}) \neq O$, we must have that $T(E_{i,j}) = \alpha E_{i,j}$ for some $\alpha \neq 0$. Thus T is one-to-one on the set of cells. ■

From this theorem, we obtain the following corollary.

COROLLARY 3.1. *If $n \geq 3$ and $T : \mathcal{M} \to \mathcal{M}$ is a linear operator that preserves sign-nonsingular matrices, then T is nonsingular.*

From Theorem 3.1 and Corollary 3.1, we have that if $n \geq 3$ and $T : \mathcal{M} \to \mathcal{M}$ is a linear operator that preserves sign-nonsingular matrices, then T is nonsingular and T preserves the set of matrices of term rank 1. Then by Theorem 2.1 (Beasley and Pullman [1, Corollary 3.1.2]) we have the following corollary.

COROLLARY 3.2. *If $n \geq 3$ and $T : \mathcal{M} \to \mathcal{M}$ is a linear operator that preserves sign-nonsingular matrices, then for any $X \in \mathcal{M}$, $T(X) = P_1(X \circ M)P_2$ or $T(X) = P_1(X \circ M)^{\mathrm{t}}P_2$, where P_1, $P_2 \in \mathcal{M}$ are permutation matrices and $M = (m_{i,j}) \in \mathcal{M}$ with $m_{i,j} \neq 0$.*

THEOREM 3.3. *If $n \geq 3$ and $T : \mathcal{M} \to \mathcal{M}$ is a linear operator, then T preserves sign-nonsingular matrices if and only if for any $X \in \mathcal{M}$, $T(X) = P_1S_1(X \circ M)S_2P_2$ or $T(X) = P_1S_1(X \circ M)^{\mathrm{t}}S_2P_2$, where $S_i \in \mathcal{M}\,(i = 1, 2)$ are diagonal matrices of ± 1's $P_i \in \mathcal{M}\,(i = 1, 2)$ are permutation matrices, and $M = (m_{i,j}) \in \mathcal{M}$ with $m_{i,j} > 0$.*

Proof. The sufficiency is easily established.

By Corollary 3.2, for any $X \in \mathcal{M}$ there exist two $n \times n$ permutation matrices P_1, P_2 and $M = (m_{i,j}) \in \mathcal{M}$ with $m_{i,j} \neq 0$ such that $T(X) = P_1(X \circ M)P_2$ or $T(X) = P_1(X \circ M)^{\mathrm{t}}P_2$. First we suppose that $T(X) = P_1(X \circ M)P_2$. Let $T_1 : \mathcal{M} \to \mathcal{M}$ be a linear operator defined by

$$T_1(X) = P_1^{\mathrm{t}}T(X)P_2^{\mathrm{t}} = P_1^{\mathrm{t}}P_1(X \circ M)P_2^{\mathrm{t}}P_2 = X \circ M$$

for any $X \in \mathcal{M}$; then clearly T_1 preserves sign-nonsingular matrices, since T preserves sign-nonsingular matrices and $P_1^{\mathrm{t}}, P_2^{\mathrm{t}}$ are permutation matrices. Now let $T_2 : \mathcal{M} \to \mathcal{M}$ be a linear operator defined by

$$T_2(X) = S_1T_1(X)S_2 = S_1(X \circ M)S_2,$$

where

$$S_1 = \operatorname{diag}\left(\frac{m_{1,1}}{|m_{1,1}|}, \frac{m_{2,1}}{|m_{2,1}|}, \ldots, \frac{m_{n,1}}{|m_{n,1}|}\right)$$

and

$$S_2 = \operatorname{diag}\left(1, \frac{m_{1,2}}{|m_{1,2}|} \cdot \frac{m_{1,1}}{|m_{1,1}|}, \ldots, \frac{m_{1,n}}{|m_{1,n}|} \cdot \frac{m_{1,1}}{|m_{1,1}|}\right).$$

Then T_2 also preserves sign-nonsingular matrices, since T_1 preserves sign-nonsingular matrices and S_1, S_2 are diagonal matrices with nonzero entries on the main diagonal. Since S_1 and S_2 are diagonal matrices, we have that

$$T_2(X) = S_1(X \circ M)S_2 = X \circ (S_1 M S_2).$$

Now let $N = S_1 M S_2$. Then $N = (n_{i,j})$ with $n_{i,j} \neq 0$. By construction of N we also have that $n_{i,1} > 0\,(1 \leq i \leq n)$ and $n_{1,j} > 0\,(1 \leq j \leq n)$. Since T_2 preserves sign-nonsingular matrices, for any sign-nonsingular matrix A we should have that $T_2(A) = A \circ (S_1 M S_2) = A \circ N$ is a sign-nonsingular matrix. If $n_{2,2} < 0$, we let

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \oplus I_{n-2},$$

then A is sign-nonsingular. Hence $A \circ N$ is a sign-nonsingular matrix. But

$$A \circ N \begin{pmatrix} n_{1,1} & n_{1,2} \\ n_{2,1} & -n_{2,2} \end{pmatrix} \oplus \operatorname{diag}(n_{3,3}, \ldots, n_{n,n})$$

is not sign-nonsingular, since

$$B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \oplus \operatorname{diag}(n_{3,3}, \ldots, n_{n,n}),$$

which has the same $(0, +, -)$ sign as $A \circ N$, is a singular matrix. This contradiction implies that $n_{2,2} > 0$. By permuting rows and columns we can prove that all entries in N are positive. Hence $N = (n_{i,j})$ with $n_{i,j} > 0$. Thus

$$T(X) = P_1(X \circ M)P_2 = P_1[X \circ (S_1^{-1} N S_2^{-1})]P_2 = P_1 S_1^{-1}(X \circ N)S_2^{-1}P_2.$$

If $T(X) = P_1(X \circ M)^{\mathrm{t}} P_2$, then by the same argument as above we can obtain that $T(X) = P_1 S_1^{-1}(X \circ M)^{\mathrm{t}} S_2^{-1} P_2$. The proof is completed. ■

We now summarize our results about the linear operators that preserve 2×2 sign-nonsingular matrices.

First, we note that T is not necessarily nonsingular. For example, let $T : \mathcal{M}_2 \to \mathcal{M}_2$ be a linear operator defined by

$$T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \quad \text{for any} \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{M}_2.$$

Then clearly T preserves sign-nonsingular matrices, since if A is a sign-nonsingular matix, then either $a \neq 0$ or $b \neq 0$, and $T(A)$ is sign-nonsingular if and only if either $a \neq 0$ or $b \neq 0$. But T is singular, since $T(E_{2,1}+E_{2,2}) = 0$. This example also tells us that there are sign-nonsingular preservers which do not strongly preserve sign-nonsigular matrices.

For another example, let $T : \mathcal{M}_2 \to \mathcal{M}_2$ be a linear operator defined by

$$T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+b & -a+b \\ a-b & a+b \end{pmatrix} \quad \text{for any} \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{M}_2.$$

If A is a sign-nonsingular matrix, then either $a \neq 0$ or $b \neq 0$. Now,

$$\det \begin{pmatrix} a+b & -a+b \\ a-b & a+b \end{pmatrix} = (a+b)^2 + (a-b)^2$$

and is zero if and only if both a and b are zero. Hence T preserves sign-nonsingular matrices, and T is a singular operator.

The proof of the theorem below is a tedious case-by-case investigation, and is omitted. For proofs of the results below, see [9].

THEOREM 3.4. *If $T : \mathcal{M}_2 \to \mathcal{M}_2$ is a linear operator that preserves sign-nonsingular matrices and T is also one-to-one on the set of cells, then T preserves the term-rank-1 matrices.*

By Theorem 2.1 (Beasley and Pullman [1, Corollary 3.1.2]) we have the following corollary:

COROLLARY 3.3. *If $T : \mathcal{M}_2 \to \mathcal{M}_2$ is a linear operator that preserves sign-nonsingular matrices and T is also one-to-one on the set of cells, then for any $X \in \mathcal{M}_2$ one has $T(X) = P_1(X \circ M)P_2$ or $T(X) = P_1(X \circ M)^{\mathrm{t}}P_2$, where $P_i \in \mathcal{M}_2\,(i = 1, 2)$ is a permutation matrix, and $M = (m_{i,j}) \in \mathcal{M}_2$ with $m_{i,j} \neq 0$.*

As in the case of $n \geq 3$, we also have:

COROLLARY 3.4. *The operator $T : \mathcal{M}_2 \to \mathcal{M}_2$ is a linear operator that preserves sign-nonsingular matrices and is also one-to-one on the set of cells if and only if for any $X \in \mathcal{M}_2$ one has $T(X) = P_1S_1(X \circ M)S_2P_2$ or $T(X) = P_1S_1(X \circ M)^{\mathrm{t}}S_2P_2$, where $P_i \in \mathcal{M}_2$ $(i = 1, 2)$ is a permutation matrix, $S_i \in \mathcal{M}_2$ $(i = 1, 2)$ is a diagonal matrix ± 1's, and $M = (m_{i,j}) \in \mathcal{M}_2$ with $m_{i,j} > 0$.*

THEOREM 3.5. *If $T : \mathcal{M}_2 \to \mathcal{M}_2$ is a linear operator that preserves sign-nonsingular matrices, then T strongly preserves sign-nonsingular ma-*

trices if and only if T is one-to-one on the set of cells.

COROLLARY 3.5. *Let $T : \mathcal{M}_2 \to \mathcal{M}_2$ be a linear operator. Then T strongly preserves sign-nonsingular matrices if and only if for any $X \in \mathcal{M}_2$ one has $T(X) = P_1 S_1 (X \circ M) S_2 P_2$ or $T(X) = P_1 S_1 (X \circ M)^t S_2 P_2$, where $P_i \in \mathcal{M}_2$ $(i = 1, 2)$ is a permutation matrix, $S_i \in \mathcal{M}_2$ $(i = 1, 2)$ is a diagonal matrix of ± 1's, and $M = (m_{i,j}) \in \mathcal{M}_2$ with $m_{i,j} > 0$.*

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